

EXISTENCE OF OUTERMOST APPARENT HORIZONS WITH PRODUCT OF SPHERES TOPOLOGY

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ABSTRACT. In this paper we find new examples of Riemannian manifolds with nonspherical apparent horizon, in dimensions four and above. More precisely, for any $n, m \geq 1$, we construct asymptotically flat, scalar flat Riemannian manifolds containing smooth outermost minimal hypersurfaces with topology $S^n \times S^{m+1}$. In the context of general relativity these hypersurfaces correspond to outermost apparent horizons of black holes.

1. INTRODUCTION AND MAIN RESULT

A well known result by Meeks, Simon and Yau [18] states that the complement of the region enclosed by the outermost minimal surface of a Riemannian 3-manifold (that is asymptotically flat –AF– with nonnegative scalar curvature) is diffeomorphic to \mathbb{R}^3 minus a finite number of balls. This is, outermost apparent horizons (in three dimensions) have spherical topology. Galloway and Schoen’s theorem [11], [10], which can be thought as a generalization of the above result, gives that apparent horizons of n -dimensional AF manifolds with nonnegative scalar curvature ($n \geq 4$) are of positive Yamabe type, i.e. admit a metric of positive scalar curvature.

By shrinking the S^{m+1} factor, it is easily seen that the product of spheres $S^n \times S^{m+1}$ has positive Yamabe invariant for all $n, m \geq 1$. A natural question that arises in view of the above discussion is to determine whether there exist examples of apparent horizons with topology $S^n \times S^{m+1}$. In this paper we address this question. Our main result is the following.

Theorem 1.1. *For any $n, m \geq 1$ there exists an asymptotically flat, scalar flat $(n + m + 2)$ -dimensional Riemannian manifold (M, g) with outermost apparent horizon which is an outermost smooth minimal hypersurface with topology $S^n \times S^{m+1}$.*

Our result also has some relevance in high dimensional relativity, since it gives suitable initial data (for the Cauchy problem) with nonspherical apparent horizon that evolves into a black hole. This behavior is particularly interesting in high dimensions due to string theory (see[19]), and also because there are the only *two* known examples of nonspherical black holes,

both specific to 5 spacetime dimensions. These are Emparan and Reall’s “rotating black ring” [8], which is a 5-dimensional spacetime with horizon topology $S^1 \times S^2$, and the “black saturn” of [7], [6], which is a black ring rotating around a spherical black hole, and was obtained by a method that also produces black rings. (In [9] there are *approximate* spacetime solutions with black ring topology $S^1 \times S^n, n \geq 2$.)

Overview of the proof. We construct (M, g) by conformally blowing up an n -sphere inside \mathbb{R}^{n+m+2} . We show that this manifold has an outermost apparent horizon because we can find barriers. Using symmetry and geometric measure theory, we prove that the horizon is a smooth minimal hypersurface that (possibly) has a conical singularity at the origin. By the maximum principle we show that the horizon has two options. (1) It is smooth everywhere and with topology $S^n \times S^{m+1}$, or (2) It is close to having (or has) a conical singularity. Finally, using an ODE analysis on the minimal surface equation we prove that a conical singularity is an attractor that causes nearby solutions to be unbounded. This rules out option (2) since the horizon is compact.

This paper is organized as follows. In §2 we introduce notation and give a general argument for the existence of apparent horizons when barriers are present. In §3 we construct (M, g) . We also write a simple formula for the mean curvature of symmetric hypersurfaces in it. We use the formula to show that (M, g) has barriers. In §4 we prove the horizon is a smooth minimal hypersurface possibly with a conical singularity at the origin. In §5 we show that the horizon is smooth everywhere and has topology $S^n \times S^{m+1}$.

2. PRELIMINARIES

Let (M^n, g) denote an n -dimensional Riemannian manifold with an asymptotically flat end E . For convenience, compactify all other ends E_k of M by adding the points $\{\infty_k\}$. A marginally-trapped region of M (sometimes called marginally outer trapped) is an open set $R \subset M$ that contains the points $\{\infty_k\}$, so that its boundary $\Sigma = \partial R$ is compact, smooth and has non-positive mean curvature. In our notation: $h_g(\Sigma) \leq 0$. The boundary Σ of the marginally-trapped region R is called a marginally-trapped hypersurface. Let \mathcal{R} denote the set of all marginally-trapped regions in M .

Following Wald [25] we define the apparent horizon of (M, g) as the boundary of the closure of the union of all marginally-trapped regions. For our purposes we are only interested in the component of the horizon that contains the end E .

Definition 2.1. The apparent horizon of (M, g) is the component Σ^* of $\partial(\overline{\cup \mathcal{R}})$ that encloses the end E .

The apparent horizon is unique and outermost since it encloses all connected marginally-trapped hypersurfaces that enclose E . On the other hand, the horizon is not necessarily smooth. Nevertheless, by symmetry and some other considerations, we will prove that, in our case, the horizon is a smooth minimal hypersurface.

A natural question that arises is to determine geometric conditions that guarantee the presence of an apparent horizon. We say that (M, g) has positive ends if it has two or more ends, and each end of M can be foliated by smooth compact hypersurfaces with positive mean curvature.

A standard argument, for which we only sketch a proof, is the following.

Theorem 2.2. *A manifold with positive ends has an apparent horizon.*

Sketch of Proof. Clearly the set \mathcal{R} is nonempty since the region bounded by a union of exactly one leaf from each end (except for the noncompactified one) is marginally trapped. We claim that no marginally trapped hypersurface can enter the positive region of the noncompact end, from which it follows that $\cup \mathcal{R} \neq M$. Indeed, suppose that a smooth compact hypersurface intersects the positive foliation of the noncompact end. At the farthest point with respect to the foliation, the hypersurface must be tangent to the foliation. By the maximum principle, the mean curvature of the hypersurface is positive at that point. Thus it is not marginally trapped. \square

Note. In 3 dimensions only spheres can be outermost apparent horizons of AF Riemannian manifolds with nonnegative scalar curvature by [18].

3. EXISTENCE

We begin this section by constructing the manifold (M, g) . Our choice of metric g is highly symmetric, as we see below. This makes the formula for the mean curvature of symmetric hypersurfaces easy to compute. Near the end of this section we use the mean curvature formula for symmetric hypersurfaces to show that M has positive ends. The existence of an apparent horizon in (M, g) follows from Theorem 2.2.

Fix $m, n \geq 1$. (M, g) is, basically, $(n+m+2)$ -dimensional Euclidean space minus an n -sphere endowed with a conformally related metric that blows up on the n -sphere.

The construction is as follows. Consider the sphere $S^n = S^n \times \{0\}^{m+1}$ sitting inside the first $(n+1)$ -dimensional factor of \mathbb{R}^{n+m+2} . Let G_p denote the Green's function for the Laplacian around $p \in \mathbb{R}^{n+m+2}$, i.e. the function $G_p(q) = |p - q|^{-(n+m)}$, where $|p - q|$ is the Euclidean norm.

For $\epsilon > 0$ (which we later require to be small) our conformal factor U is the smooth positive function defined on $\mathbb{R}^{n+m+2} \setminus S^n$ by the formula

$$(1) \quad U(p) = 1 + \epsilon G_p * \chi_{S^n} = 1 + \epsilon \int_{S^n} |p - q|^{-(n+m)} d\mu(q).$$

Definition 3.1. We define (M, g) to be $\mathbb{R}^{n+m+2} \setminus S^n$ endowed with the conformally flat metric $g = U^{4/(n+m)} \delta_{ij}$. (For simplicity, ϵ is removed from the definition of g .)

Lemma 3.2. *For (M, g) defined as above we have that*

- (i) *g is $SO(n+1) \times SO(m+1)$ invariant, so the group acts by isometries on (M, g) and fixes both ends.*
- (ii) *$\Delta_0 U = 0$ outside S^n , so (M, g) is scalar flat.*
- (iii) *Let $r(p) = \text{dist}(p, S^n)$. For p near S^n we have $U(p) = 1 + \epsilon r^{-m} + O(r^{1-m})$ whenever $m > 1$. For $m = 1$, $U(p) = 1 + \epsilon r^{-1} + O(\log r)$.*
- (iv) *There exists $\alpha = \alpha(n, m) > 0$ so that as $|p| \rightarrow \infty$, $U(p) = 1 + \epsilon \alpha |p|^{-(n+m)} + O(|p|^{1-(n+m)})$, so spatial infinity is asymptotically flat.*

Proof. (i) is direct from the definition of U . For (ii), Green's formula gives that U is harmonic. The transformation law for the scalar curvature under conformal deformations is $R_g = -U^{-(n+m+4)/(n+m)} (4(n+m+1)/(n+m) \Delta_0 U - R_0 U)$ which implies $R_g = 0$. (iii) follows from the expansion $U(p) = 1 + \epsilon \int_{S^n} (p_1^2 + \dots + p_{m+1}^2 + \sum_{i=1}^{n+1} (p_{i+m+1} - \xi_i)^2)^{-(n+m)/2} d\mu(\xi)$, which is a particular case of the calculation in the Appendix of [20]. The asymptotic formula in (iv) holds because of the maximum principle together with $u(p) \rightarrow 1$ as $|p| \rightarrow \infty$, and $u(p) \rightarrow +\infty$ as $\text{dist}(p, S^n) \rightarrow 0$. Standard calculations as those in [1] show that the end is asymptotically flat in this case. \square

The manifold (M, g) is invariant under the action of $SO(n+1) \times SO(m+1)$. We are interested in finding a formula for the mean curvature of $SO(n+1) \times SO(m+1)$ -invariant hypersurfaces of (M, g) since the apparent should also be invariant. In order to do that, we first find a formula for the mean curvature of $SO(n+1) \times SO(m+1)$ -invariant hypersurfaces of Euclidean space, and then apply the transformation law for the mean curvature under conformal deformations.

Let $\Sigma \subset (\mathbb{R}^{n+m+2}, \delta_{ij})$ be an $SO(n+1) \times SO(m+1)$ -invariant hypersurface. If we write a vector in $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ as (x, y) respecting this decomposition, then Σ has the form

$$(2) \quad \Sigma(\gamma) = \{(x, y) : (|x|, |y|) \in \gamma\},$$

where γ is some curve in the quadrant $\mathbb{R}_+^2 = \{(a, b) \geq (0, 0)\}$.

If $\gamma = (x(s), y(s))$ is oriented and parametrized by arc length, the principal curvatures of $\Sigma(\gamma)$ with respect to an outward pointing unit normal are $\kappa, -\dot{x}/y, \dots, -\dot{x}/y, \dot{y}/x, \dots, \dot{y}/x$, where κ is the curvature of γ with respect to the flat metric. Therefore, the mean curvature of Σ inside Euclidean space is given by

$$(3) \quad h_0 = \kappa + n\dot{y}/x - m\dot{x}/y.$$

Example 3.3. The coordinate sphere in \mathbb{R}^{n+m+2} is $S_R(0) = \Sigma(\gamma)$, where $\gamma = (R \cos s/R, R \sin s/R)$, $0 \leq s \leq \pi R/2$. (3) gives $h_0 = (n + m + 1)/R$.

The transformation law for the mean curvature under conformal deformations gives that the mean curvature of a hypersurface $N \subset (M, g)$ is

$$(4) \quad h = U^{2-c} (h_0 + c\partial_\nu \ln U),$$

where $c = 2(n + m + 1)/(n + m)$ and ∂_ν is the Euclidean outward-pointing normal derivative.

If we consider an $SO(n + 1) \times SO(m + 1)$ -invariant hypersurface $\Sigma(\gamma)$ like before, and fix the orientation of γ so that the outward-pointing normal derivative is given by $\partial_\nu = \dot{y}\partial_x - \dot{x}\partial_y$, then the quantity (4) from above is easily computed using equation (3). We write this in the following result.

Lemma 3.4. *Let $\gamma = (x(s), y(s))$ be parametrized by arc length and oriented as above. Then, the mean curvature of $\Sigma(\gamma) \subset (M, g)$ is*

$$(5) \quad h = U^{2-c} \left(\kappa + \dot{y} \left(\frac{n}{x} + c \frac{U_x}{U} \right) - \dot{x} \left(\frac{m}{y} + c \frac{U_y}{U} \right) \right),$$

where $c = 2(n + m + 1)/(n + m)$. If γ is the graph of $y(x)$, the mean curvature is given by

$$(6) \quad h = \pm \frac{U^{2-c}}{\sqrt{1 + y'^2}} \left(\frac{y''}{1 + y'^2} + y' \left(\frac{n}{x} + c \frac{U_x}{U} \right) - \left(\frac{m}{y} + c \frac{U_y}{U} \right) \right).$$

A discussion on minimal hypersurfaces of the Euclidean metric that are $SO(n+1) \times SO(m+1)$ -invariant appears in [17]. Radially symmetric minimal hypersurfaces of Euclidean space have been studied in [13], [14] and [24]. Singular solutions of the minimal surface equation in flat space appear in [2], where minimal cones in dimension 8 are constructed as examples of minimizers that are singular on the origin. More recent references include [4] and [23].

We now use equation (5) to compute the mean curvature of barriers that foliate the ends of M . The foliations consist of large coordinate spheres on the asymptotically flat end, and small tubes around the missing S^n on the other end. An essential fact is that the tubes around the missing S^n have topology $S^n \times S^{m+1}$ since they are of the form $\partial(S^n \times B^{m+2})$.

Theorem 3.5. *M has positive ends.*

Proof. We begin by foliating a neighborhood of the missing S^n by the tubes $T_r = \Sigma(\gamma_r)$, where $\gamma_r = (1 + r \cos(s/r), r \sin(s/r))$ for $r > 0$ small, $s \in [0, \pi r]$. Using the asymptotics of Lemma 3.2 (iii) and plugging γ_r into (5) gives

$$h(T_r) = -U^{2-c} \left(\frac{1}{r} \left[(m-1) - \frac{2m(n+m+1)}{(n+m)(1+r^m/\epsilon)} \right] + \frac{n \cos(s/r)}{1+r \cos(s/r)} \right) \\ + \text{lower order terms,}$$

which is a positive quantity for $0 < r < \epsilon^{1/m}$ and $\epsilon > 0$ small enough. (For this calculation we reverse the sign of equation (5) since the outward-pointing normal of the tubes T_r points towards the missing sphere.)

On the other hand, the mean curvature of all large enough coordinate spheres $S_R(0) = \Sigma(\gamma_R)$, where $\gamma_R = (R \cos(s/R), R \sin(s/R))$ for $t \in [0, \pi R/2]$ is positive since M is asymptotically flat. Indeed, it follows from example 3.3, equation (5) and Lemma 3.2 (iv) that $h(S_R) = O(1)((n+m+1)/R - \epsilon \cdot O(R^{-(n+m+2)}))$. The right-hand side of this equation is positive for all R large enough, and continues to be so for all small values of $\epsilon > 0$. \square

Corollary 3.6. *M has an apparent horizon.*

Remark 3.7. During a conversation with Hubert Bray we realized that U defined as above makes the quantity $h(T_r)$ of Theorem 3.5 be positive. This observation motivated much of our work.

4. REGULARITY

In this section we prove that the horizon is a smooth minimal hypersurface except possibly at the origin, where it may have a conical singularity. This last option is ruled out in the next section.

Recall that given a curve $\gamma \subset \mathbb{R}_+^2$, the set $\Sigma(\gamma)$ is defined as the subset $\Sigma(\gamma) = \{(|\vec{x}|, |\vec{y}|) \in \gamma\} \subset \mathbb{R}^{n+m+2}$. We will see that the apparent horizon is given by $\Sigma(\gamma^*)$, where γ^* is a particular curve. As a matter of fact, from what we have done so far we can already say a few things about γ^* , assuming this decomposition holds. For example, the horizon is contained in a compact set, therefore γ^* should be contained in a bounded region. The horizon encloses the missing sphere, therefore γ^* should separate \mathbb{R}_+^2 in two regions, one containing all the tubular barriers of Theorem 3.5. This motivates the following definition. Let $N_\epsilon \subset \mathbb{R}_+^2$ denote a fixed (depends only on ϵ) semicircular marginally trapped region that is a neighborhood of $(1, 0)$. (This is possible by the proof of Theorem 3.5.)

Definition 4.1. Let $\gamma : (-L_1, L_2) \rightarrow \mathbb{R}_+^2$, $0 \leq L_1, L_2 \leq \infty$ be a curve parametrized by arc length. We say that γ is minimal if it is smooth,

$h(\Sigma(\gamma)) = 0$, and $\text{int}(\mathbb{R}_+^2) - \gamma$ has two components, one of them being bounded and containing N_ϵ from above.

Theorem 4.2 (Interior regularity). *The apparent horizon is $\Sigma^* = \Sigma(\gamma^*)$, where γ^* (called the horizon curve) is a minimal curve.*

Proof. Since (M, g) is $SO(n+1) \times SO(m+1)$ -invariant it follows that the horizon also is. Therefore, there is a rectifiable curve γ^* with $\Sigma^* = \Sigma(\gamma^*)$, and it follows that γ^* is bounded since the horizon is compact.

Claim. γ^* is smooth in the interior of \mathbb{R}_+^2 .

Indeed, the apparent horizon outer-minimizes area in the homology class of a marginally trapped hypersurface. Since the mean curvature of the marginally trapped hypersurface is nonpositive, a standard argument shows that the horizon is actually a minimizing current. A well known result of geometric measure theory (see e.g. [22]) is that the singular set of minimizing currents has codimension 7 or greater. On the other hand, singularities of γ^* in the interior of \mathbb{R}_+^2 translate into codimension 2 singularities of the horizon $\Sigma^* = \Sigma(\gamma^*)$, which proves the claim.

To finalize, recall that a minimizing current is stationary, so wherever the horizon is smooth it must be a minimal hypersurface. \square

Remark 4.3. Whenever $m + n + 2 \leq 7$ the above theorem gives that the horizon is a smooth minimal hypersurface everywhere. If $m, n \leq 6$, the horizon may only have a conical singularity at the origin. Indeed, since the horizon is minimizing it has codimension 7 singularities. If $n + m + 2 \leq 7$, the singular set of the horizon can only be empty. For the other case, note that by symmetry the singular set can only have dimension 1, n or m . This way, if $n, m \leq 6$, the singular set may only be 1-dimensional, so it must be a conical singularity at the origin (see Theorem 4.6 below).

From equation (5) it follows that the condition $h(\Sigma(\gamma)) = 0$ for minimal curves translates into the following equation for the curve $\gamma(s) = (x(s), y(s))$:

$$(7) \quad \kappa + \dot{y} \left(\frac{n}{x} + c \frac{U_x}{U} \right) - \dot{x} \left(\frac{m}{y} + c \frac{U_y}{U} \right) = 0 \quad \text{on } \text{int}(\mathbb{R}_+^2).$$

Solutions to this equation are graphical almost everywhere with respect to both axes.

Lemma 4.4. *Except at isolated vertical (resp. horizontal) points, a curve that solves (7) is locally a graph over the x -axis (resp. y -axis).*

Proof. Suppose $\dot{x}(s) = 0$ for all $s \in (s_0, s_1)$. Writing equation (7) for $s_0 < s < s_1$ gives $n/x_0 + cU_x/U = 0$, where $x_0 = x((s_1 + s_0)/2)$. This is impossible since U_x/U is never constant on vertical segments. A similar argument works for the horizontal case. \square

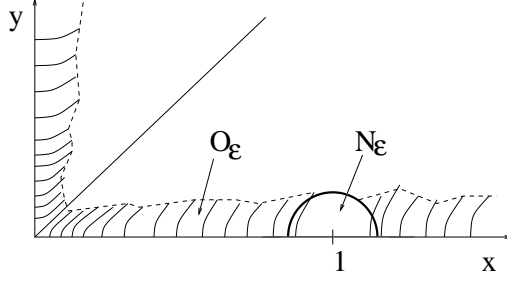


Figure 1. The neighborhood $O_\epsilon \subset \mathbb{R}_+^2 - N_\epsilon$ is foliated by solutions of (7).

We will show that the apparent horizon is a minimal hypersurface smooth everywhere except possibly at the origin, where it may have a conical singularity. (We remove this last possibility in the next section.) In order to prove this statement we need the following ODE result. Recall that $N_\epsilon \subset \mathbb{R}_+^2$ is the marginally trapped neighborhood of $(0, 1)$ from before.

Proposition 4.5. *Smooth solutions of equation (7) with an endpoint on the axes minus N_ϵ exist uniquely. This is, if γ is smooth and solves equation (7) with $\gamma(0) \in (\text{axes} - N_\epsilon)$, then γ is unique. Furthermore, $\dot{\gamma}(0)$ is perpendicular to the axis, unless $\gamma(0) = (0, 0)$, in which case the angle is $\tan^{-1} \sqrt{m/n}$. It follows that there exists a neighborhood O_ϵ of the axes inside $\mathbb{R}_+^2 - N_\epsilon$ that is foliated by these unique solutions. (See Figure 1.)*

Proof. From Lemma 4.4, solutions of equation (7) with an endpoint on the y -axis are graphical over the x -axis for a short while. In that case, equation (7) becomes

$$(8) \quad y''/(1+y'^2) + y'(n/x + cU_x/U) - (m/y + cU_y/U) = 0.$$

An ODE analysis of this equation gives local existence of smooth solutions. By inspecting the equation it follows that the condition $y'(0) = 0$ (or $y'(0) = \sqrt{m/n}$ when $y(0) = (0, 0)$) is required. The same can be done for solutions that intersect the x -axis outside N_ϵ . \square

Using the ODE proposition we prove the main result of this section.

Theorem 4.6 (Regularity). *The apparent horizon of (M, g) is a minimal hypersurface smooth everywhere except possibly at the origin, where it may have a conical singularity.*

Proof. The main idea of the proof is to show that if γ is a minimal curve that does not match one of the smooth solution curves that foliate O_ϵ (from Proposition 4.5) as it approaches the boundary, then $\Sigma(\gamma)$ cannot be an area minimizer. We will show this for a curve that has an endpoint in the y -axis.

The analysis for endpoints on the x -axis (outside N_ϵ) is analogous.

Let $\gamma = (x, y) : (-L_1, L_2) \rightarrow \mathbb{R}_+^2$ be a minimal curve. Without loss of generality we analyze the behavior at L_2 only, where we have $\liminf_{s \rightarrow L_2} x(s) = 0$.

Claim 1. If $\Sigma(\gamma)$ outer-minimizes area in the homology class of a barrier, then $L_2 < \infty$, $y(s), x(s)$ are eventually monotonic as $s \rightarrow L_2$ and extend continuously to $(-L_1, L_2]$

Indeed, if $\dot{y}(s_i) = 0$ for infinitely many $s_i \rightarrow L_2$, then the curve must be tangent to some leaf of the foliation of O_ϵ . By uniqueness of second order ODEs it follows that γ becomes the leaf itself. This gives that y is eventually monotonic and that $\lim_{s \rightarrow L_2} y(s)$ exists and is finite. Now suppose $x(s)$ has infinitely many local minima as $s \rightarrow L_2$, which are arbitrarily close to the y -axis. By cutting off the curve at one of these points (that is close enough to the axis) and replacing that segment by a horizontal line up to the axis, the length of the curve decreases. Furthermore, the cutoff curve continues to enclose the barrier and the area of $\Sigma(\gamma)$ decreases (since the area form is $x^n y^m U^{2c} ds$). Thus we have found a curve enclosing the barrier with less area than the horizon –but this is impossible since $\Sigma(\gamma)$ outer-minimizes. We deduce that both $x(s), y(s)$ are eventually monotonic. Since the curve is bounded and parametrized by arc length, it follows that L_2 must be finite. This way, the curve extends continuously to $(-L_1, L_2]$ since it is Lipschitz. From the above argument it follows that $L_1, L_2 < \infty$ and γ is continuous on $[0, L]$, where $L = L_1 + L_2$. This proves the claim.

Claim 2. γ meets the axes perpendicularly.

From Lemma 4.4, γ is locally a graph. In particular, around its endpoint on the y -axis, γ is the graph of $y(x)$ over the interval $[0, \delta)$ for some $\delta > 0$. Assume initially that $y(0) > 0$. An analysis of equation (8) gives that $y'(0)$ exists and is either zero or \pm infinity. If $y'(0) = \pm\infty$ then γ cannot minimize by a cut-off argument like the one above. This way, it follows that $\Sigma(\gamma)$ is smooth so long as γ does not go through the origin. Nevertheless, in that case, we have that

Claim 3. If $\gamma(0) = (0, 0)$ then γ meets the origin at an angle $\tan^{-1} \sqrt{m/n}$.

To prove this last part we make a blow up argument around the origin. Consider the rescaled curve $\gamma_\lambda = (u(s), v(s)) = (\lambda x(s/\lambda), \lambda y(s/\lambda))$. Multiplying equation (7) by $1/\lambda$ and evaluating at s/λ , we get that, in the limit $\lambda \rightarrow \infty$, the rescaled curve solves $\kappa + n\dot{v}/u - m\dot{u}/v = 0$. Since this equation is invariant under rescaling of the graph, its solution must be a line. This way γ has a tangent line at the origin. A direct calculation shows that this line has an angle $\tan^{-1} \sqrt{m/n}$.

It is evident that if $\gamma(0) = (0, 0)$ the apparent horizon has a singularity at the origin. By the rescaling argument from above we get that the tangent

cone of the horizon at the origin is the $SO(n+1) \times SO(m+1)$ -invariant minimal cone in \mathbb{R}^{n+m+2} . This ends the proof. \square

5. TOPOLOGY

Our goal here is to prove the following result.

Theorem 5.1. *The apparent horizon is smooth everywhere and has topology $S^n \times S^{m+1}$.*

Recall that, from before, the horizon is given by $\Sigma(\gamma^*)$. Furthermore, the horizon is everywhere smooth, unless γ^* goes through the origin –in that case $\Sigma(\gamma^*)$ gets a conical singularity around the origin.

Idea of Proof of Theorem 5.1. The main idea of the proof is to prove that the curve γ^* is (roughly speaking) a small semicircle around the point $(1, 0)$. This way, γ^* does not go through the origin, and the horizon does not have a conical singularity. Furthermore, we get that the topology of the horizon is automatically $S^n \times S^{m+1}$.

In order to show that γ^* is “semicircular” we prove that there are three regions in the plane that γ^* (including its endpoints) must avoid altogether. Please see Figure 2 for a depiction of them.

Region (1). The marginally trapped region N_ϵ which is a small enough semicircle around $(1, 0)$. This region is already forbidden since the end is positive. It exists by the argument in the proof of Theorem 3.5.

Region (2). The region $\{y > c'\epsilon^{1/(n+m)}\} \subset \mathbb{R}_+^2$. This region is forbidden by the maximum principle which we prove in Subsection 5.1 below. (Here $c' > 0$ will be a constant independent of ϵ .)

Region (3). A small portion of the axes around the origin. We rule out this region in Subsection 5.2 below. This portion is forbidden because there is a conical attractor that makes “almost conical curves” become complete graphs, which are unbounded.

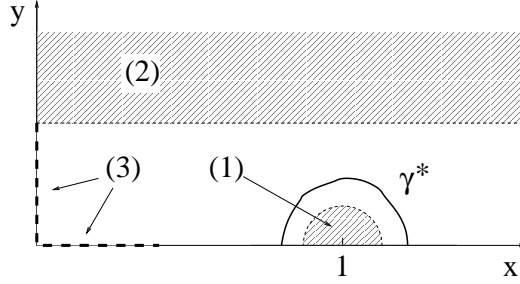
It follows that once regions (1)–(3) are forbidden, all that γ^* can be is a deformation of a semicircle around $(1, 0)$. This is because γ^* is simple, bounded, and intersects the x -axis on each side of $(1, 0)$.

5.1. Maximum principle. We begin by finding an estimate for the derivatives of U which will be used in the maximum principle.

Lemma 5.2. *Let $r^2 = (x-1)^2 + y^2$ as before. Then*

$$(9) \quad |U_x| \leq \epsilon c' r^{-(m+n+1)} \quad \text{and} \quad 0 \leq -U_y \leq \epsilon c' r^{-(n+m+1)},$$

where $c' > 0$ is a constant independent of ϵ .

Figure 2. The horizon curve γ^* and the forbidden regions.

Proof. From the expansion $U(x, y) = 1 + \epsilon \int_{S^n} (y^2 + (x - \xi_1)^2 + \xi_2^2 + \dots + \xi_{n+1}^2)^{-(n+m)/2} d\mu(\xi)$ it follows that $|U_x| \leq \epsilon c'(x+1)r^{-(m+n+2)}$ and that $0 \leq -U_y \leq \epsilon c'yr^{-(n+m+2)}$, which gives the result. \square

Theorem 5.3 (Maximum principle). γ^* is contained in $\{y \leq c'\epsilon^{1/(n+m)}\}$, where $c' > 0$ is a constant independent of ϵ .

Proof. We will show that the maximum height of γ^* is bounded by $c'\epsilon^{1/(n+m)}$. Indeed, let $\gamma^* = (x(s), y(s))$. From the proof of Theorem 4.6 it follows that $y(s)$ has a global maximum at some $s_0 > 0$ for which $\dot{y}(s_0) = 0$. We first claim that this global maximum is not located on the y -axis. Indeed, this follows from the estimates of Lemma 5.10 below, which show that the curve is concave up around the y -axis. If the curve is oriented so that $s = 0$ at the furthest endpoint on the x -axis, then at s_0 we have $k(s_0) \geq 0$ and $\dot{x}(s_0) = -1$. Evaluating equation (7) at s_0 gives $\kappa(s_0) + m/y(s_0) + cU_y/U = 0$. Using the estimate of Lemma 5.2 together with $U \geq 1$ and $\kappa \geq 0$ we get $m/y(s_0) \leq \epsilon c'r^{-(n+m+1)}$. Since $y \leq r$, it follows that $y(s_0)^{n+m} \leq cc'\epsilon/m$. This ends the proof. \square

5.2. The conical attractor. Here we set $\delta = \epsilon^{1/(n+m+1)}$ to make some expressions simpler. We are interested in proving the following result.

Theorem 5.4. For $\delta > 0$ small enough, any curve that solves equation (7) and has an endpoint on $\{0\} \times [0, \delta]$ or on $[0, \delta] \times \{0\}$ is a complete graph, hence unbounded.

We will give a proof of this theorem for curves that have one endpoint on $\{0\} \times [0, \delta]$. The other case is analogous.

Remark 5.5. The depiction of Figure 2 is accurate. Indeed, $(\{0\} \times [0, \delta]) \cup ([0, \delta] \times \{0\})$ intersects $\{y > c'\epsilon^{1/(n+m)}\}$ of the maximum principle since $\delta = \epsilon^{1/(n+m+1)}$ and $\epsilon^{1/(n+m+1)} > c'\epsilon^{1/(n+m)}$ for $\epsilon > 0$ small enough.

In order to proceed with the proof, we introduce a change of coordinates in equation (8) to study its behavior. Similar coordinates were used by Ilmanen in [17] to study wiggly companions of minimal cones, which satisfy equation (8) with $U \equiv 1$. In that case, the change of coordinates makes the equation an autonomous two-dimensional system, for which stability analysis is simple. In our case, our equations are not autonomous so the analysis is more involved.

Consider the coordinates $t = \log x$, $W = y/x$, $Z = dy/dx$. Using the chain rule, equation (8) becomes the first order system

$$(10) \quad W_t(t, W, Z) = Z - W, \quad Z_t(t, W, Z) = n \frac{1 + Z^2}{W} (K_1 - K_2 W Z),$$

where $K_1(t, W) = m/n + cyU_y/(nU)$ and $K_2(t, W) = 1 + cyU_x/U$.

Let γ be a curve that solves equation (7) and has an endpoint on $\{0\} \times [0, \delta]$. Lemma 4.4 gives that γ is the graph of a function $y(x)$ that solves equation (8) near the axis. Therefore, we may use the (W, Z) coordinates and equation (10) to obtain the “phase” of γ , i.e. the tuple (W, Z) that represents γ inside of WZ -space (at least for a short time) which is given by the above change of coordinates.

Idea of the proof of Theorem 5.4. Equation (10) is “almost” autonomous, which is the motivation for examining the phase plane in the (W, Z) coordinates. (In the Euclidean case, the analogous equation becomes autonomous. See Figure 3(a), and [17].) We show that there exists a trapping region Ω in the plane, for which the evaluation of the vector field (W_t, Z_t) on $\partial\Omega$ always points towards its interior (See Figure 3(b)). Furthermore, we prove that Ω is bounded in the $Z(= dy/dx)$ -direction. This gives that any curve whose phase enters Ω does not blow up in finite time, therefore it is a complete graph and is unbounded. In the last part of the proof we find global estimates that show that the phase of curves that solve equation (7) and have an endpoint on $\{0\} \times [0, \delta]$ eventually lies in Ω . We can expect this behavior since, in the Euclidean case, a minimal hypersurface that is close to a minimal cone at the origin converges to the cone at infinity. In particular, it remains graphical and is unbounded.

We now introduce the conical attractor, which actually is just an attracting region. (In the Euclidean case, curves in phase space do spiral into the attracting fixed point given by the minimal cone.)

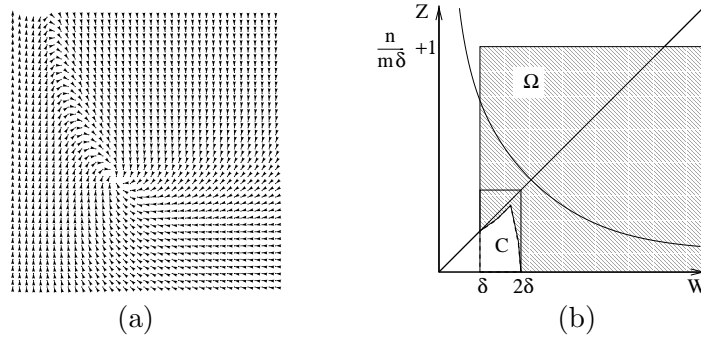


Figure 3. (a) Direction field (Z_t, W_t) for $U = 1$. (b) The trapping region Ω .

Proposition 5.6. *Let $\Omega' = \{(\delta, 0) \leq (W, Z) \leq (+\infty, n/(m\delta) + 1)\}$ and C as below. Then $\Omega = \Omega' - C$ is a trapping region, i.e. the quantity (W_t, Z_t) of equation (10) evaluated on $\partial\Omega$ always points towards the inside of Ω .*

The region C is a region bounded by two special curves and two lines. Roughly speaking, C is a “worst case scenario region” for the vector field, as we see below. See Figure 3(b) for a depiction of all of them.

We begin by bounding the terms K_1, K_2 from above. We will later use these bounds to construct the worst case scenario region C .

Lemma 5.7. *Let K_1, K_2 be as in the system (10). On the region $\{(t, W) : W \geq \delta, t \in \mathbb{R}\}$ we have the uniform bounds*

$$\frac{m}{n} - c'\epsilon^\alpha \leq K_1(t, W) \leq \frac{m}{n}, \quad \text{and} \quad 1 - c'\epsilon^\alpha \leq K_2(t, W) \leq 1 + c'\epsilon^\alpha,$$

where $\alpha = 1 - (m + n)/(m + n + 1) > 0$.

Proof. $W \geq \delta$ is just $y \geq \delta x$, so it must be $r \geq \delta$ in that region. The rest follows from applying the estimate (9). \square

A consequence of this estimates are global bounds for the vector field (W_t, Z_t) in a particular region of the WZ -plane.

Corollary 5.8. *Let $R = [\delta, 2\delta] \times [0, 2\delta]$ and (W_t, Z_t) from (10). Then*

$$\inf_{\mathbb{R} \times R} W_t \geq -2\delta, \quad \text{and} \quad \inf_{\mathbb{R} \times R} Z_t \geq \frac{m - nc'\epsilon^\alpha}{2\delta} - \delta(1 + c'\epsilon^\alpha)(2n + 8\delta^2),$$

where $\alpha = 1 - (m + n)/(m + n + 1) > 0$.

The set C is constructed as follows. Let σ denote an integral curve of (W_t, Z_t) in the WZ -plane that starts at the bottom right corner of R , i.e. at the point $p = (2\delta, 0)$. (Note: such a curve exists by general ODE results, but it is not necessarily unique since the system (10) is not autonomous.) Let $R_1(\sigma) \subset R$ denote the region inside R that is bounded on top by the

diagonal and bounded on the right by the curve σ . We define C_1 to be the intersection of all the possible $R_1(\sigma)$, where σ is an integral curve that starts at p . (This is, C_1 is a worst case scenario region for integral curves that start at p .) C_1 is not empty. Indeed, a direct calculation using Corollary 5.8 shows that

Lemma 5.9. *Let L be the line $Z(W) = -W(m/4\delta^2) + m/2\delta^2$. Then C_1 includes the region $R_1(L)$, and $R_1(L) \supset (\{\delta\} \times [0, \delta] \cup [\delta, 2\delta] \times \{0\})$.*

Now let σ denote an integral curve that starts at $q = (\delta, \delta)$, and let $R_2(\sigma)$ denote the region inside C_1 bounded on top by the curve σ . Note that any such σ is increasing (as a graph on the W -axis) since W_t is positive inside R . We define C to be the intersection of all the possible $R_2(\sigma)$, where σ is an integral curve that starts at q . (This is, add to C_1 a worst case scenario for integral curves that start at q .) C is not empty since σ is increasing. Indeed, we are, at worst, capping off the top of C_1 by a horizontal line through q . See Figure 3(b).

Proof of Proposition 5.6. The boundary of Ω consists of five parts as depicted in Figure 3(b). The roof is given by a segment of the line $\{Z = n/(m\delta) + 1\}$ that lies well inside the region $Z = K_1/(K_2W)$. This gives that Z_t points downward on the top. The bottom boundary is part of the positive W -axis, and Z_t is positive there. The left-side boundary consists of three segments. One is the segment of the line $\{W = \delta\}$ that lies above the diagonal, for which W_t is nonnegative. What is left is the top and right of the boundary of C . From the construction it follows that (W_t, Z_t) points inwards there as well. \square

We now show that the phase of any curve that solves equation (8) and has an endpoint on $\{0\} \times [0, \delta]$ eventually lies in Ω . In order to do that, we compute a second order approximation of these solutions. We check that after a small time they are close to the conical solution. (See Figure 1 for a snapshot of this phenomena.)

Lemma 5.10. *A smooth solution of equation (8) with $y(0) > 0$ satisfies*

$$y''(0) = \frac{m}{(n+1)} \frac{1}{y(0)} + c \frac{U_y/U - U_x/U}{n+1} = \frac{m}{(n+1)} \frac{1}{y(0)} + \epsilon O(r^{-(m+n+1)}).$$

Proof. We know from the proof of Theorem 4.6 that $y'(0) = 0$. Multiplying (8) by y , taking limit $x \rightarrow 0$ and using L'Hopital's rule in the term y'/x gives $(n+1)y''(0)y(0) - m + y(0)c(U_x/U - U_y/U) = 0$. Together with the gradient estimate of Lemma 5.2 and the fact that $U \geq 1$ this gives the result. \square

Corollary 5.11. *Suppose y solves equation (8) with $y(0) = \delta$ for $\delta > 0$ small enough. Then for $0 \leq x \leq \delta$, we have that*

$$y(x) = \delta + \frac{m}{2(n+1)\delta}x^2 + O(\epsilon), \quad \text{and} \quad y'(x) = \frac{mx}{(n+1)\delta} + O(\epsilon).$$

Proof. Follows from Lemma 5.10 and Taylor's theorem. \square

Proposition 5.12. *For $\delta \geq 0$ small enough, the phase of any solution y of equation (8) with $y(0) = \delta$ eventually enters the region Ω .*

Proof. We first prove this for a curve with $y(0) = 0$. Indeed, in that case $y(x) = \sqrt{m/n}x + O(\epsilon)$ and $y'(x) = \sqrt{m/n} + O(\epsilon)$, from before. This way, for small $x > 0$ we get $W = y/x \approx \sqrt{m/n}$ and $Z = y' \approx \sqrt{m/n}$, so (W, Z) lies inside Ω .

Whenever $y(0) = \delta > 0$ is small, using the expansions of Corollary 5.11 evaluated at $x = \delta$ give $W = y(\delta)/\delta \approx 1 + m/2(n+1)$ and $Z = y'(\delta) \approx m/(n+1)$, which also gives a point that lies within Ω . \square

Proof of Theorem 5.4. Follows directly from the above proposition together with Proposition 5.6. This ends the proof of Theorem 1.1 as well. \square

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